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The buoyant motion within a hot gas plume in a horizontal wind

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A smoke plume that has become nearly horizontal at some distance from the source behaves much like a 'line-thermal', for which, using a perturbation method, a solution in laminar flow may be obtained, on the supposition that excess temperatures are small and buoyant movements slow, i.e. that the Rayleigh number of the problem is suitably low. In analogy with some other problems in turbulent flow and turbulent diffusion, the laminar solution is then assumed to approximate what is observed in the turbulent case, provided that the rate of growth of the diffusing cloud is assessed realistically.

The so-calculated pattern of streamlines in a cross-section of the plume agrees qualitatively with the observed behaviour of hot plumes and puffs, consisting of two vortex-like structures of opposite sense of rotation, lying on either side of the plume centre. The bodily upward movement of the plume is found to depend critically on the rate of growth of the plume. Thus when the plume diameter grows faster than linearly with distance (such behaviour characterizes the 'quasi-asymptotic' stage of relative diffusion predicted by Batchelor, 1952, for which Richardson's (4/3)-power law of eddy diffusivity holds) the plume tends to reach an asymptotic height. A crude theoretical estimate of the asymptotic height attained shows fair agreement with observations reported elsewhere. Although the plume nearly reaches this asymptotic height in the quasi-asymptotic phase, it retains a small gradient in the final phase which may be of importance at large distances from the source. The small-Rayleigh-number criterion restricts the validity of the solution to 'weakly buoyant' plumes.

1. Introduction

The problem of dispersal of pollutants in the atmosphere requires a study of the buoyant motion in smoke plumes because most major sources of dust and gaseous pollutants discharge their effluents at a temperature considerably above that of the atmospheric air. The usual engineering approach to this aspect of the problem is to assume an 'effective' chimney height at which the smoke plume is supposed to turn horizontal and then calculate the diffusion of this horizontal cloud. The difference between the 'effective' and the physical chimney heights is regarded as due to the upward momentum and the buoyancy of the emitted smoke. The magnitude of this difference is also determined by the speed and the 'gustiness' of the wind. It is, moreover, not difficult to show that, for such important sources as power station chimneys, buoyancy is far more important than initial momentum in determining the 'effective' chimney height, except in so far as 'downwash' is concerned (Hawkins & Nonhebel 1955).

Reduced to its simplest form then, the problem is to elucidate the behaviour of a buoyant mass of fluid, injected at a constant rate into a moderately strong horizontal wind. Attempts to provide a theoretical solution have been made by Bosanquet, Carey & Halton (1950), Sutton (1950), Priestley (1956) and Scorer (1959). Sutton (1950) and Priestley (1956) start by considering a vertically rising plume and assume that this is simply 'sheared over' by the wind, so that the various horizontal cross-sections are stacked on one another at an inclination $\tan^{-1}(w/U)$, where w is the velocity of upward drift and U the wind velocity. Sutton proceeds by adopting the 'mixing length' hypothesis while Priestley assumes Gaussian temperature and velocity profiles. At distances from the chimney large compared with its diameter, both theories predict an asymptotic mean plume path of the form

$$z = \text{const.}\, x^n, \tag{A}$$

where z is the height above the chimney top, x the horizontal distance along wind, and n an exponent having the value of $\frac{2}{3}$ in Sutton's treatment, $\frac{3}{4}$ in Priestley's. In fact if one plots the mean path of almost any plume within the first 300 m of the chimney one finds that an equation of the form (A) fits the data moderately well, the value of the exponent n lying between 0·5 and 1·0, depending on the kind of source and on atmospheric conditions. Scorer (1959) also finds a result of the form (A) with $n = \frac{2}{3}$ (as Sutton's exponent) on the basis of dimensional reasoning.

All these results appear to depend strongly on the asumption that the diameter of the growing plume is proportional to z, the height reached above the chimney top. This assumption, in turn, may be justified by the two hypotheses that

(1) the distribution of velocity, etc., across the jet is 'self-preserving', and

(2) the effects of ambient turbulence are negligible.

Clearly, the second of these hypotheses can only hold relatively close to the chimney where the buoyant jet's self-created turbulence is the main diffusing agent. For this phase of plume motion it is indeed plausible to transfer laboratory results on the spreading of jets, as has been done by Bosanquet *et al.* (1950). However, it is intuitively apparent that when the velocity of the plume relative to the surrounding air mass becomes small compared to wind speed (i.e. where the inclination of the plume against the horizontal becomes small) the environmental turbulence takes over as the main dispersing agency. There the 'spreading' problem becomes one of turbulent diffusion relative to the plume's centre of mass, in a field of turbulence which in a first approximation may be assumed to be homogeneous.

That the environmental turbulence becomes important at a certain stage of development for the plume and for isolated thermals has been recognized by Priestley (1956) and also by Turner (1963). These authors introduce a 'second phase' of plume behaviour in which a loss of heat to the surroundings is postulated in a manner which is not very clear. The turbulent motions of the atmosphere are supposed to diffuse the cloud in the 'first phase' but in the 'second phase' they also partly remove its heat and momentum from what may be regarded as the effective plume. It would be perhaps more correct to say that the plume enters a phase of more vigorous diffusion, but the consequences of this are not worked out by Priestley in a way that could be practically applied. It is indeed clear both from theory and observation that after an initial phase of relatively slow diffusion, in which the plume has a regular outline, it enters a régime of accelerated spreading and 'break-up'.

A further reason why the theories of Sutton and Priestley have limited applicability is that when the inclination of the plume is of the order of 10° it is very difficult to accept the proposition that a vertical plume has simply been 'sheared over' by the wind. A more realistic view of most smoke plumes is to regard them as nearly horizontal, having a slow upward drift caused by buoyancy. The approach of Bosanquet *et al.* (1950) is based essentially on this model and uses mixing length assumptions, similar to those adopted by Sutton (1950), but with the difference that the plume diameter is now regarded as proportional to x, the downwind distance from the source, rather than to z, the vertical distance. The asymptotic form of the result obtained is then

$$z = \text{const.} \log x. \tag{B}$$

The basic hypothesis underlying equation (B) is still that the plume's self-created turbulence is the main diffusing agent, as in a jet exhausting into a quiescent atmosphere. At a large enough distance from the source a smoke plume no longer grows linearly with distance and there the theory of Bosanquet *et al.* becomes invalid.

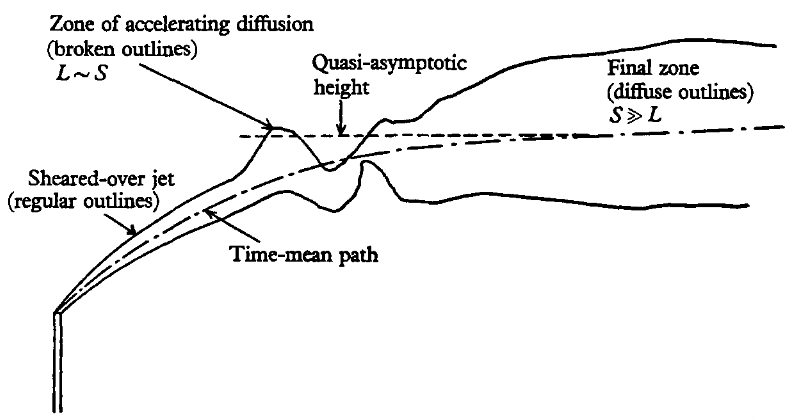

FIGURE 1. Phases in the spreading and upward drift of a hot smoke plume in a horizontal wind.

These considerations suggest, then, that at least in calculating the 'effective' chimney height another phase of plume behaviour should be considered in which the transport of heat and momentum is carried out by the environmental turbulence, assumed uniform and homogeneous for simplicity. This phase, in turn, may be subdivided into two régimes on the basis of Batchelor's (1952) results on relative diffusion. While the size of the plume is within the Kolmogoroff range of inertial eddies, the rate of growth increases with plume size (zone of accelerated diffusion). Further downwind, where the plume is large compared to the scale of turbulence, the rate of spread is slower, the plume diameter grows with the square root of time or distance drifted downwind, as in molecular diffusion (final zone).

The above described scheme of buoyant plume behaviour is illustrated in figure 1. Essentially the same conclusions have been reached previously (Csanady 1961) on the basis of some visual and photographic field observations. The theory to be developed below shows that in the zone of accelerated diffusion (in the Kolmogoroff range, that is) the plume tends to a quasi-asymptotic height. Before completely reaching this, however, accelerated diffusion comes to an end and the plume ultimately enters its final phase, in which theory predicts a small, constant slope for the mean path. In practical applications the quasi-asymptotic height could perhaps serve as the 'effective chimney height', for the slope in the final phase is ordinarily very small indeed.

A limitation of the theory as developed here is that it only applies to 'weakly buoyant' plumes, meaning those drifting upward with a velocity less than the r.m.s. turbulent velocities. When this is not the case the path of the plume becomes a more complex function, but even then the theory at least earmarks the important non-dimensional variables.

Apart from elucidating the behaviour of the plume as regards its mean path, the present treatment also yields detailed information on the flow pattern of drift velocities within the plume. Two vortex-like structures of opposite sense of rotation are found to lie parallel to the plume axis, on a horizontal diameter, roughly at the edges of the thermal plume. These vortex-like structures have repeatedly been observed in plumes and thermals (Scorer 1958; Turner 1960, 1963; Lilly 1962, 1964; Keffer & Baines 1963) and some crude theories have been proposed to explain their existence. In fact the most elementary application of continuity leads one to expect a downflow region somewhere outside the rising plume; the present treatment provides some information on the details, which may be useful in certain applications such as fall-out of dust from a hot plume.

The discussion above refers to a plume rising in a dynamically 'neutral' atmosphere. The effect on the plume path of small departures from the neutral gradient (as would normally occur in well-stirred layers) is likely to be negligible as may be judged from some calculations of Bosanquet *et al.* (1950). It should be noted, however, that the treatment to be given below is capable of extension to the non-neutral case. Another limitation of the theory as developed below is that it treats a plume far above ground but this may also be removed by a further development of the theory.

2. Slow buoyant movements in a fluid, caused by an instantaneous line source of heat

As a prelude to later discussion we shall develop here a theory of slow buoyant movements (in laminar flow) which would take place in a fluid at rest, following the instantaneous release of heat along the x-axis. Immediately after release there would be some fast movements, but in time as the heat released spread to a large mass of fluid, the excess temperatures would become small, the buoyant movements slow. At this stage it seems reasonable to describe the motion by linearized equations, neglecting squares and products of velocities and excess temperatures. Morton (1960) has similarly treated the axisymmetric case introducing for the solution an expression in terms of the Rayleigh number governing the problem. The solution below corresponds to the first term in Morton's expansion.

The linearized equations expressing conservation of mass, momentum and energy are:

$$\left.\begin{aligned} &\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\\ &\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v,\\ &\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w+g\frac{\theta}{T_a},\\ &\frac{\partial \theta}{\partial t}=k\nabla^2\theta. \end{aligned}\right\} \tag{1}$$

Here T_a is the ambient temperature, θ temperature excess, g is the acceleration of gravity, k is thermal diffusivity, ∇^2 is the two-dimensional Laplacian operator, the rest of the notation being conventional. A stream function is conveniently introduced,

$$v=\partial\psi/\partial z,\quad w=-\partial\psi/\partial y. \tag{2}$$

Another useful variable to work with is the x component of vorticity

$$\xi=\nabla^2\psi. \tag{3}$$

The relevant solution of the energy equation is well known

$$\theta=\theta_0\exp\{-(y^2+z^2)/4kt\}. \tag{4}$$

The centre temperature θ_0 depends on the heat liberated initially, Q, per unit length of the x-axis

$$\theta_0=Q/4\pi\rho c_p kt. \tag{5}$$

In place of heat release it is convenient to introduce a new variable

$$F=gQ/\pi\rho c_p T_a. \tag{6}$$

The dimensions of this variable are length^3/time^2.

By eliminating pressure from equations (1) one obtains the vorticity equation

$$\frac{\partial\xi}{\partial t}=\nu\nabla^2\xi-\frac{g}{T_a}\frac{\partial\theta}{\partial y}. \tag{7}$$

In view of equations (4) to (6) the buoyancy term in the last result may be written as

$$\sigma(y,z,t)=-\frac{g}{T_a}\frac{\partial\theta}{\partial y}=\frac{yF}{2S^4}\exp\left(-\frac{y^2+z^2}{4kt}\right). \tag{8}$$

Equation (7) is the heat conduction equation with a source term. In the absence of boundaries its solution is

$$\xi=\int_0^t\frac{dt'}{4\pi\nu(t-t')}\iint_{-\infty}^{\infty}\sigma(y',z',t')\exp\left\{-\frac{(y-y')^2+(z-z')^2}{4\nu(t-t')}\right\}dy'\,dz'. \tag{9}$$

For our purposes it will be sufficient to treat the algebraically simple case of $\nu = k$ (Prandtl number unity). For this case, substituting σ from equation (8) and carrying out the integrations one finds

$$\xi = \frac{Fy}{8k^2t}\exp\left(-\frac{y^2+z^2}{4kt}\right). \tag{10}$$

This result may be put into a non-dimensional form by introducing the following scales:

$$\text{length,}\quad S = (2kt)^{\frac{1}{2}};\qquad \text{velocity,}\quad C = F/2k. \tag{11}$$

The relevant non-dimensional variables are then

$$\xi^* = \xi S/C,\quad y^* = y/S,\quad z^* = z/S.$$

Equation (10) may hence be written as

$$\xi^* = \tfrac{1}{2}y^*\exp\{-\tfrac{1}{2}(y^{*2}+z^{*2})\}. \tag{12}$$

In this last result the time no longer appears explicitly: the vorticity distribution remains self-similar, although its scale grows with the square root of time. A similar conclusion holds, of course, for the temperature distribution (see equation (4)) with the qualification that the absolute value of the temperature excess also decreases with t^{-1}.

Having found the vorticity distribution the flow problem is in principle solved. To compute the corresponding flow pattern one must, however, solve equation (3). Far from boundaries, and in view of the symmetry of the problem, one may write the required solution in polar co-ordinates as

$$\psi^*(r,\phi) = f(r)\cos\phi. \tag{13}$$

Here $\psi^* = \psi/SC$ is the non-dimensional stream function. Substituting (12) and (13) into (3) one arrives at the ordinary differential equation (dropping stars from the non-dimensional quantities)

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{1}{r^2}f = \tfrac{1}{2}r\,e^{-\frac{1}{2}r^2}. \tag{14}$$

This has the following well-behaved integral

$$f(r) = -\tfrac{1}{2}r^{-1}(1-e^{-\frac{1}{2}r^2}). \tag{15}$$

The homogeneous equation has the solutions r and r^{-1}, both of which are ruled out by the boundary conditions that w is finite at the origin and zero at infinity. Thus the non-dimensional stream-function is given by

$$\psi = -(\cos\phi/2r)(1-e^{-\frac{1}{2}r^2}). \tag{16}$$

The flow pattern specified by this function is shown in figure 2. Vertical velocities along the horizontal diameter across the centre of the plume are given by (in a non-dimensional form)

$$w^* = -\partial\psi/\partial r = \tfrac{1}{2}\{e^{-\frac{1}{2}r^2}-(1-e^{-\frac{1}{2}r^2})/r^2\}. \tag{17}$$

The value of this at $r = 0$ is $\frac{1}{4}$, so that the vertical velocity at the plume centre is

$$w_0 = \tfrac{1}{4}C, \tag{18}$$

having returned to dimensional quantities.

The distribution of the non-dimensional velocity w^* along the horizontal diameter across the plume centre is shown in figure 3. Downward flow is found to occur outside $r^* = 1{\cdot}6$. One may say that a vortex-like structure is centred at $r^* = 1{\cdot}6$, where the mean excess temperature is 28 % of the excess temperature at the plume centre. In other words, the eye of the vortex is close to the edge of the thermal plume.

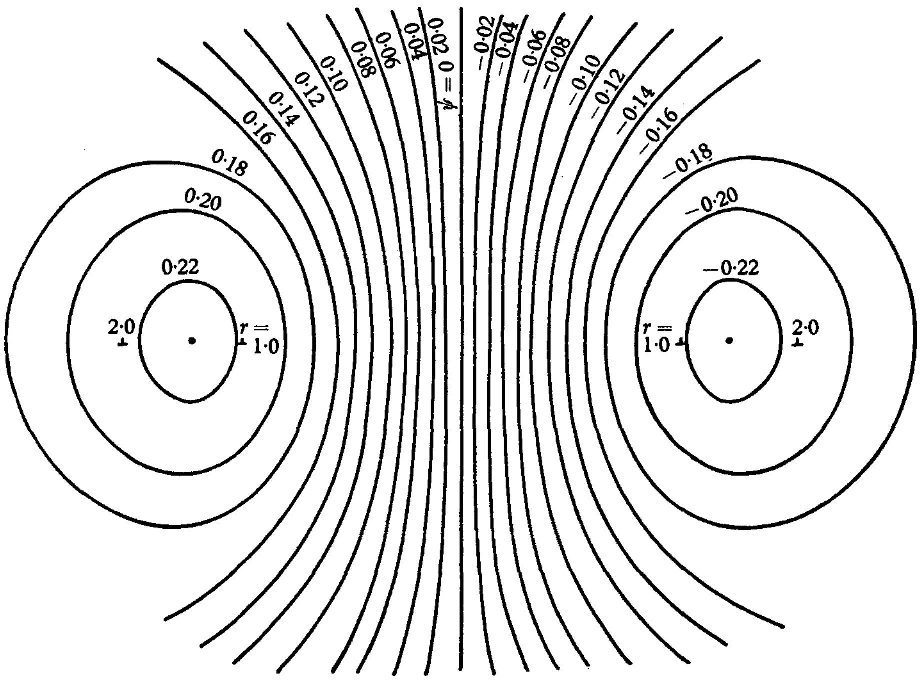

FIGURE 2. Cross-section of flow pattern in a weak line-thermal.

In two recent papers Lilly (1962, 1964) has reported on the numerical simulation of buoyant convection induced by a line-source of heat. The problem is thus identical with the one discussed above, although Lilly's approach is different in that he is able to retain the non-linear terms in the equations of motion and energy, since he works with the aid of an electronic computer. His resulting flow patterns show considerable similarity to the one calculated here.

3. Application of the theory to turbulent flow

The introduction of eddy exchange coefficients for heat, matter and momentum was an early and obvious attempt to describe turbulent transport of these entities. With increasing knowledge of the structure of turbulent flow it has become clear that this simple model is certainly incorrect in detail: many of the turbulent eddies are too large for their contribution to the flux of heat, etc., to be set proportional to the local gradient. Townsend (1956, p. 107) discusses these questions in some detail.

Nevertheless, it is also a fact that the overall features of observed temperature or concentration distributions show a remarkable resemblance to those characterizing molecular diffusion, with the single important difference that the rate of growth of, say, a diffusing cloud is several orders of magnitude greater when turbulence is present. In particular, molecular diffusion from a point source leads to a Gaussian distribution of temperature or concentration while in the presence of turbulence the same functional form of mean quantity distribution is approached to an approximation adequate in practice. Similarly, velocity profiles in a turbulent two-dimensional boundary layer or mixing layer can be described quite well with the aid of the laminar solutions (Townsend 1956; Clauser 1956).

These experimental facts must somehow be reconciled with conclusive theoretical and experimental evidence to the effect that turbulent transport cannot possibly be proportional to a local gradient. One is forced to conclude that molecular and turbulent diffusion have some deep-lying fundamental similarities, which do not, however, extend to the ordinary linear coupling of 'fluxes' and 'forces', to use the language of irreversible thermodynamics. At this point one may recall the statistical model of random walk, from which the classical diffusion equation may be derived by proceeding to the limit of very short steps. Clearly, it is not *necessary* to fall back on a differential equation; this is only done for mathematical convenience in finding solutions. The Gaussian distribution follows directly from the random-walk model. If one regards the latter a crude but essentially realistic model of turbulent (as well as molecular) diffusion, the situation becomes considerably clearer. It is then not surprising that both types of diffusion lead to the same (or very similar) distributions, in particular a Gaussian distribution if there is a point source. Almost incidentally, in the case of molecular diffusion it is possible to proceed to the limit of vanishing steps and define local transport coefficients. That in turbulent flow this is not possible is a relatively unimportant distinction physically. Starting with the premise that the point-source distribution is Gaussian, it is possible to define 'virtual' eddy diffusivities (Batchelor 1949, 1952) which would be those necessary to produce the same distributions in molecular diffusion, without any pretence of dealing with a local transport coefficient. The above rather vague arguments may be summarized in the hypothesis:

'The mechanism of molecular and turbulent diffusion being fundamentally similar, it is possible to approximate the spatial distribution of matter in turbulent diffusion by the distribution applying in molecular diffusion under identical boundary and initial conditions, provided that the different rate of growth is allowed for.' It should be emphasized once more that this is logically *not* the same step as assuming transport to be proportional to gradient, even if the end result is very much the same.

The whole argument of this section applies to 'relative' diffusion in turbulence as well as to diffusion observed by fixed-point instruments, even though the experimental evidence for the normality of distribution in relative diffusion is rather meagre. In relative diffusion the effective steps in the random-walk process would be only those which separate particles, not those which all particles of the cloud take together ($\equiv$ those caused by eddies large compared to cloud size). Another difficulty to be discussed before the approach of the previous section is used in dealing with turbulent flow is that the random-walk model does not apply directly to the transfer of momentum or vorticity. If an expression similar to (9) is to hold in turbulent flow, vorticity should behave like a transferable scalar property. From the mean-value vorticity equation (Townsend 1956, p. 28) one may deduce that ξ would indeed behave in this manner if the correlation $\overline{\xi'(\partial u'/\partial x)}$ were zero. There is in fact no physical reason why, say, a positive value of ξ' should predominantly be associated with a positive gradient $\partial u'/\partial x$. If one accepts that this correlation is indeed negligible, it becomes reasonable to describe the changes in the mean vorticity component ξ as those corresponding to a 'random walk' of the vorticity which is convected without change by the turbulent movements. A very similar argument holds for the spread of vorticity in a two-dimensional turbulent boundary layer and it may be appropriate to refer once more to the work of Clauser (1956) who demonstrated that the behaviour of the 'outer' part of such a layer can be well understood with the aid of laminar solutions.

According to these considerations we shall now assume that the solution of the previous section applies to the growth and buoyant movement of a line-thermal in turbulent surroundings, provided that its rate of growth is assessed realistically. To the accuracy of this approximation (relying on the random-walk model) the spread of heat and vorticity takes place identically. This justifies *a posteriori* the assumption $k = \nu$ made above. The standard deviation S of the Gaussian distribution of heat or vorticity is given in molecular diffusion by

$$S^2 = 2kt.$$

In turbulent diffusion, if we start with the premise of a Gaussian distribution, the classical heat-conduction equation may be shown to hold (Batchelor 1949), provided that

$$dS^2/dt = 2k_T, \tag{19}$$

where k_T is the virtual eddy diffusivity.

To proceed further it is necessary to introduce some assumption regarding the functional form of $S(x)$ or $k_T(x)$, or another relation connecting these two quantities. As pointed out before, the dilution of a smoke plume is a problem in 'relative' diffusion (Batchelor 1950, 1952) in which the dispersal of particles about their centre of mass is relevant. One well-known feature of relative diffusion is that the rate of separation of particles is greater when they are farther apart. Batchelor (1952) explains this by remarking that the range of eddy sizes contributing to the relative velocity is larger when two particles are farther apart. Both this theory and the observations of Richardson (1926) show the eddy diffusivity to be proportional to $S^{\frac{4}{3}}$. For dimensional reasons the relationship in the *inertial subrange* must be of the form

$$k_T = \beta\epsilon^{\frac{1}{3}}S^{\frac{4}{3}}, \tag{20}$$

where β is a constant of order unity. On integrating equation (19) one finds then

$$S = \epsilon^{\frac{1}{2}}[\tfrac{2}{3}\beta(t-t_0)]^{\frac{3}{2}}, \tag{21}$$

$$k_T = \tfrac{2}{3}\beta^3\epsilon(t-t_0)^2. \tag{22}$$

Here t_0 is an 'effective origin', related to the conditions of release. There are two other régimes of relative diffusion. Close to the source Batchelor (1952) finds

$$dS^2/dt = \text{const.}\, t. \tag{23}$$

This is equivalent to $S \sim t$ and $k_T \sim t$ which are the well-known similarity laws for a jet.

In the final phase of diffusion, where the particles wander independently in a very large plume their standard deviation about their centre of mass will be equal to their standard deviation from a fixed source. From Taylor's (1922) theory of diffusion by continuous movements one has thus

$$dS^2/dt = \text{const.} = 2k_T. \tag{24}$$

At this stage one should examine the conditions of validity of the perturbation analysis of the previous section. The typical length is S, typical velocity C (equation (11)), hence the condition that inertia forces are negligible compared to 'eddy-viscous' forces is

$$SC/k_T < Re_l, \tag{25}$$

where Re_l is a limiting Reynolds number, above which inertia forces must be taken into account. Substituting the definition of C this becomes

$$FS/2k_T^2 < Ra_l = Re_l, \tag{26}$$

which shows that the limiting Reynolds number of the buoyant movements is at once a limiting Rayleigh number. A substitution of the values of S and k_T for the initial and intermediate phase shows the Rayleigh number decreasing rapidly. In the intermediate phase, in particular,

$$Ra = FS/2k_T^2 = \text{const.}\, F/\epsilon^{\frac{3}{2}} t^{\frac{5}{2}}.$$

Introducing a dissipation length L by $\epsilon = u^3/L$, where u is the r.m.s. turbulent velocity, one may rewrite the Rayleigh number as

$$Ra = \text{const.}\, (F/u^2L)\, (L/ut)^{2\cdot 5}.$$

In the final phase, on the other hand,

$$Ra = \text{const.}\, (F/u^2L)\, (ut/L)^{0\cdot 5} = \text{const.}\, (F/u^2L)\, (S/L).$$

Thus when the diffusing cloud becomes large enough compared to the scale of turbulence, the Rayleigh number will exceed the limiting value and the perturbation approach breaks down. Just when this occurs depends on the value of the non-dimensional parameter F/u^2L: the perturbation analysis will hold as long as the intensity of turbulence is suitably high

$$u^2 > \text{const.}\, F/L. \tag{27}$$

The value of the constant in this formula could only be determined by experiment, but judging by other evidence on slow, viscosity dominated motion, it should be of order unity, provided that $S \leqslant L$. In physical terms, equation (27) restricts the validity of the perturbation analysis to *weak* line-thermals, so that the velocity characterizing buoyant movements would be smaller than the r.m.s. turbulent velocity. Even with such weak thermals, however, the first-order theory breaks down at a certain large enough size.

4. Application to chimney plumes

A chimney may be idealized as a continuous point source of heat placed into a steady and uniform wind of speed U, on which there is superimposed a homogeneous field turbulence, of intensity u^2 and scale L. Focusing on conditions some distance away from the chimney, one may expect that the temperature excess is already small enough, the buoyant motion slow enough for the linearized equations of motion to hold. These equations are then easily seen to be identical with equations (1) of §2, except that the time-gradient $\partial/\partial t$ is replaced by $U\partial/\partial x$. The only other departure from the theory developed in §2 occurs in the dimensions of the heat release, Q: instead of heat per unit length, we have now heat released per unit time. The new variable F of equation (6) (now referred to as 'flux of buoyancy') will have the changed dimension (length^4/time^3). In order that the same non-dimensional equations be obtained it is therefore necessary to modify the definition of the velocity-scale C (equation (11)), and write

$$C = F/2k_T U. \tag{28}$$

Physically, one may argue that a quantity of heat δQ discharged during time-interval δt will be contained in a cylinder of length $\delta x = U\delta t$; hence the replacement of F of the line-thermal by F/U of the plume. (Note that F is still defined by equation (6), with Q = the strength of the source in cal/sec or equivalent units.) With this modification the solution developed in §2 is valid for a cross-section of a 'weakly buoyant' smoke plume. The flow pattern has been displayed as figures 2 and 3 in a non-dimensional form. This applies without change, the velocity scale being given by equation (28), the length-scale by the expressions of the previous section appropriately modified.

The required modification consists of replacing the time of growth t in equations (19) to (24) by the time of travel x/U of a given plume-section. Thus, for example, (19) becomes

$$k_T = \tfrac{1}{2}U(dS^2/dx), \tag{19a}$$

and so on. With the changed definition of C the small Rayleigh number criterion (26) becomes

$$Ra = FS/2Uk_T^2 < Ra_1. \tag{26a}$$

In the intermediate phase the Rayleigh number becomes

$$Ra = \text{const.}\,(F/u^2UL)\,(UL/ux)^{2\cdot5}.$$

In the final phase,
$$Ra = \text{const.}\,(F/u^2UL)\,(ux/UL)^{0\cdot5}.$$

At this stage it is convenient to introduce the non-dimensional variable of 'gustiness', $G = u/U$. The non-dimensional distance variable $D = x/L$, also suggests itself. A third non-dimensional combination occurring in the above expression of the Rayleigh number is

$$\phi = F/U^3L.$$

The combination F/U^3 has been used before as the relevant length scale for the thermal rise of the plume in a discussion of observed data (Csanady 1961). The Rayleigh number may now be written

$$Ra = \text{const.}\,\phi G^{-4\cdot5}D^{-2\cdot5} \quad \text{(intermediate phase)},$$
$$Ra = \text{const.}\,\phi G^{-1\cdot5}D^{0\cdot5} \quad \text{(final phase)},$$

with the constant being theoretically of order unity. The perturbation theory holds as long as the Rayleigh number is suitably low. This criterion (equation (27)) may also be expressed as

$$G^2 > \text{const.}\,\phi, \tag{27a}$$

again provided that the plume is not too large ($S \leqslant L$), the constant being presumably of order unity. At higher Rayleigh numbers the solution could be written down in principle as an expansion in increasing powers of Ra (Morton 1960), so that it would still be a function of the non-dimensional parameters ϕ, G and D.

Assuming that the Rayleigh number is low enough for the perturbation theory to hold, it is possible to express the mean path of the plume in terms of the non-dimensional parameters just defined. Although an integral from zero to infinity of the upward velocity along the horizontal diameter across the plume centre

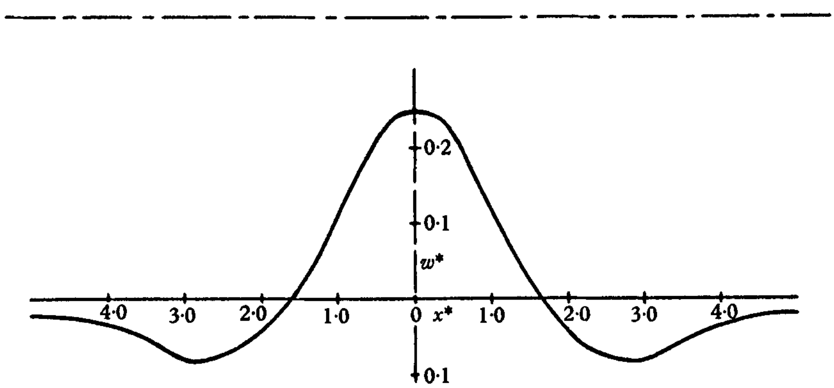

FIGURE 3. Velocity profile along horizontal diameter.

yields exactly zero upward transport (in figure 3, the positive and negative areas are equal, in figure 2 the $\psi = 0$ streamline returns at infinity) within the thermal plume itself there is a predominantly upward flow with an average vertical velocity somewhat less than the centre velocity. One may say that the bodily upward drift of the plume has a velocity

$$dh/dt = w_a = \alpha C, \tag{29}$$

where the constant α has the order of magnitude of 10^{-1} (cf. figure 3). Introducing the definition of C from (28) and writing $t = x/U$ one obtains

$$dh/dx = \alpha F/2U^2k_T. \tag{30}$$

Observing that $k_T = k_T(x)$ this represents a differential equation for the mean path of the plume. In the *initial* stage one obtains

$$dh/dx = \text{const.}/x \quad \text{or} \quad h = \text{const.}\log x \quad \text{(initial stage)},$$

which is equivalent to equation (B) of the introduction, representing the Bosanquet *et al.* (1950) result. This is, of course, a consequence of the assumption that $S = \text{const.}\,x$ near the origin (which is the same as the similarity law for a jet), also used by Bosanquet *et al.* The present theory has been constructed to apply at larger distances from the source so that more interest attaches to the intermediate phase, where

$$\frac{dh}{dx} = \frac{9\alpha}{8\beta^3} \frac{F}{\epsilon(x-x_0)^2},$$

which integrates to

$$h - h_1 = \frac{9\alpha F}{8\beta^3 \epsilon}\left[\frac{1}{x_1 - x_0} - \frac{1}{x - x_0}\right]. \tag{31}$$

Here h_1 is the height of the plume at the commencement of the intermediate phase and x_1 is another 'effective origin' marking the commencement of the intermediate phase (figure 4). The difference between the two effective origins, $(x_1 - x_0)$, could be regarded a characteristic time scale of the turbulence, t_d, times wind speed

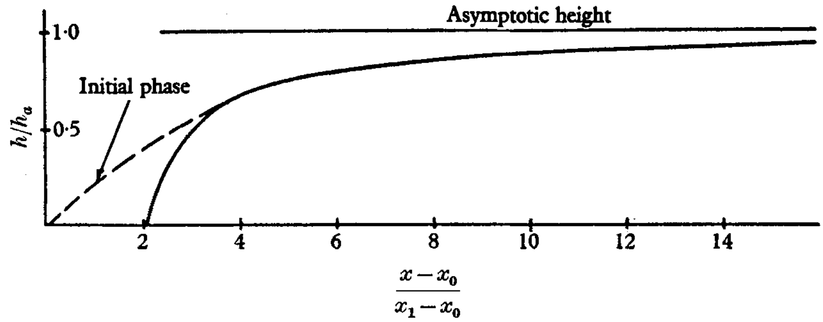

FIGURE 4. Schematic representation of the mean path of a smoke plume.

$$x_1 - x_0 = Ut_d.$$

Experimentally, the time scale t_d could be estimated from the growth of a puff of smoke, initially small. At time t_d after release the puff would have grown big enough for its diffusion to be governed by eddies in the inertial subrange. The onset of this phase of accelerated diffusion may be identified with the visually observed 'break up' of the puff into several distinct portions, in marked contrast to its original, fairly smooth expansion. On dimensional grounds the only reasonable hypothesis is

$$t_d = \varphi L/u,$$

where φ is a constant of order unity. One has now, for the asymptotic plume height (as $x \to \infty$) from equation (31),

$$h_a - h_1 = \zeta F/U^3, \tag{32}$$

where $\zeta = 9\alpha/8\beta^3\varphi G^2$ contains the two constants β and φ vaguely described before as 'of order unity'. If one assumes that these could be anywhere between 0·5 and 2·0, $\alpha = 0{\cdot}1$ and $G = 0{\cdot}05$ to 0·20 (an order of magnitude estimate of gustiness) one finds ζ to be between 0·3 and 360. This is a rather uncertain result and it is difficult to compare with experiment because equation (32) refers to the intermediate phase only. However, if we assume that initial conditions (chimney diameter, for example) do not affect the asymptotic height, a similar formula may be written down by dimensional reasoning for the *entire* thermal rise,

$$h_a = \zeta F/U^3, \tag{32a}$$

where, however, ζ would depend in an unknown way on the parameters governing atmospheric turbulence.

Crude observations at Tallawarra, New South Wales (Csanady 1961) have indicated a value $\zeta = 250$. This is consistent with the theoretical estimate, especially if one observes that the latter only accounts for a certain fraction of ζ. It is to be noted that the contribution from the intermediate phase is inversely proportional to the *square* of gustiness, although this rests on the rather uncertain estimate of the distance $(x_1 - x_0)$.

Further downstream the plume enters the final phase of relative diffusion in which by Taylor's (1922) theorem

$$dS^2/dx = GL,$$

with L a length scale equal in order of magnitude to the dissipation length. Consequently,

$$dh/dx = \alpha F/GLU^3 = \text{const.}$$

Using the asymptotic height of the intermediate phase from equation (32) one may express this last result as

$$dh/dx = \alpha h_a/\zeta GL. \tag{33}$$

In typical practical situations h_a is of the same order of magnitude as L (50 m, say) so that if $G = 0{\cdot}1$, the gradient dh/dx in the final phase of diffusion will be of the order of 10^{-2}. This is quite small, but when one considers the movement of the plume over several km a rate of rise of 10 m/km may be of importance.

The results of this section may be combined with those of an earlier paper (Csanady 1961) to give the following comprehensive picture of plume behaviour. When the effects of initial upward momentum are comparatively minor, the significant length scale of upward drift is

$$l = F/U^3,$$

where F is 'flux of buoyancy' and U wind speed. Close to the source the plume rises according to

$$h/l = K(x/l)^n,$$

where the constant K has a value between 1·7 and 2·2 (depending on the characteristics of atmospheric turbulence), and the constant n lies between $\frac{2}{3}$ and $\frac{3}{4}$ as given by Sutton (1950), Priestley (1956) and Scorer (1959). At some distance from the source, typically of the order of 300 m, the plume 'breaks up' and enters a more vigorous phase of diffusion. In this phase it almost reaches an asymptotic height given by

$$h_a/l = \zeta,$$

where the constant ζ has, in an observed case, the value of 250 and is probably proportional to the inverse second power of gustiness. This asymptotic height is nearly reached at distances typically of the order of 1000 m. On a scale of several km it may be important that the plume does not quite turn horizontal, but in its final stage retains a gradient of the order of magnitude

$$dh/dx = \alpha l/GL,$$

where L is the dissipation scale of turbulence, G is gustiness and α a constant approximately equal to 0·1.

All the above results pertain to a neutral atmosphere, a moderately strong horizontal wind and a plume far removed from the ground. When the plume grows very large, the results are subject to the further qualification that the perturbation theory as developed here ceases to apply and a 'Rayleigh number effect' may become evident. In the present application this means that the mean path of the plume may become a more complex function of the form

$$\frac{h}{L} = f\left(\frac{x}{L}, \frac{F}{U^3L}, \frac{u}{U}\right).$$

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